AN OPERATOR EXTENSION OF THE PARALLELOGRAM LAW AND RELATED NORM INEQUALITIES

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Dedicated to my teacher Aziz Atai Langroudi with respect and affection

ABSTRACT. We establish a general operator parallelogram law concerning a characterization of inner product spaces, get an operator extension of Bohr's inequality and present several norm inequalities. More precisely, let $\mathfrak{A}$ be a C^*-algebra, T be a locally compact Hausdorff space equipped with a Radon measure μ and let $(A_t)_{t\in T}$ be a continuous field of operators in $\mathfrak{A}$ such that the function $t \mapsto A_t$ is norm continuous on T and the function $t \mapsto \|A_t\|$ is integrable. If $\alpha : T \times T \to \mathbb{C}$ is a measurable function such that $\overline{\alpha(t,s)}\alpha(s,t) = 1$ for all $t, s \in T$, then we show that

$$\int_T\int_T |\alpha(t,s)A_t - \alpha(s,t)A_s|^2\, d\mu(t)d\mu(s) + \int_T\int_T |\alpha(t,s)B_t - \alpha(s,t)B_s|^2\, d\mu(t)d\mu(s)$$
$$= 2\int_T\int_T |\alpha(t,s)A_t - \alpha(s,t)B_s|^2\, d\mu(t)d\mu(s) - 2\left|\int_T (A_t - B_t)d\mu(t)\right|^2 .$$

1. INTRODUCTION

Let $\mathfrak{A}$ be a C^*-algebra and let T be a locally compact Hausdorff space. A field $(A_t)_{t\in T}$ of operators in $\mathfrak{A}$ is called a continuous field of operators if the function $t \mapsto A_t$ is norm continuous on T. If $\mu(t)$ is a Radon measure on T and the function $t \mapsto \|A_t\|$ is integrable, one can form the Bochner integral $\int_T A_t \mathrm{d}\mu(t)$, which is the unique element in $\mathfrak{A}$ such that

$$\varphi\left(\int_T A_t \mathrm{d}\mu(t)\right) = \int_T \varphi(A_t)\mathrm{d}\mu(t)$$

for every linear functional φ in the norm dual $\mathfrak{A}^*$ of $\mathfrak{A}$; see [7, Section 4.1] and [6].

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