AUTOMATIC CONTINUITY OF HIGHER DERIVATIONS

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ABSTRACT. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras. A sequence $\{d_n\}$ of linear mappings from $\mathcal{A}$ into $\mathcal{B}$ is called a higher derivation if $d_n(a_1a_2) = \sum_{k=0}^{n} d_k(a_1)d_{n-k}(a_2)$ for each $a_1, a_2 \in \mathcal{A}$ and each nonnegative integer n. In this paper, we show that if $\{d_n\}$ is a higher derivation from $\mathcal{A}$ into $\mathcal{B}$ such that d_0 is onto and $\ker(d_0) \subseteq \ker(d_n)$ $(n \in \mathbb{N})$, then there is a sequence $\{\delta_n\}$ of derivations on $\mathcal{B}$ such that

$$d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \ldots + r_i} \right) \delta_{r_1} \ldots \delta_{r_i} d_0 \right).$$

As a corollary we prove that a higher derivation $\{d_n\}$ from a Banach algebra into a semisimple Banach algebra is continuous provided that d_0 is onto and $\ker(d_0) \subseteq \ker(d_n)$ $(n \in \mathbb{N})$. We also deduce that if $\mathcal{A}$ is a semisimple Jordan Banach algebra and $\{d_n\}$ is a higher derivation on $\mathcal{A}$ with $d_0(\mathcal{A}) = \mathcal{A}$ and $\ker(d_0) \subseteq \ker(d_n)$ $(n \in \mathbb{N})$ then $\{d_n\}$ is continuous.

1. INTRODUCTION

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras, $\mathcal{X}$ be a $\mathcal{B}$-bimodule and $\sigma : \mathcal{A} \to \mathcal{B}$ be a linear mapping. A linear mapping $\delta : \mathcal{A} \to \mathcal{X}$ is called a σ-derivation if it satisfies the generalized Leibniz rule $\delta(a_1a_2) = \delta(a_1)\sigma(a_2) + \sigma(a_1)\delta(a_2)$ for each $a_1, a_2 \in \mathcal{A}$. In the case $\mathcal{A} = \mathcal{B} = \mathcal{X}$ and $\sigma = I_{\mathcal{A}}$, the identity mapping on $\mathcal{A}$, a σ-derivation is called a derivation. (For other approaches to generalized derivations and their applications see [2, 3, 5, 14, 13] and references therein. In particular, an automatic continuity problem for (σ, τ)-derivations is considered in [12] and an achievement of continuity of (σ, τ)-derivations without linearity is given in [7].)

A sequence $\{d_n\}$ of linear mappings from $\mathcal{A}$ into $\mathcal{B}$ is called a higher derivation if $d_n(a_1a_2) = \sum_{k=0}^{n} d_k(a_1)d_{n-k}(a_2)$ for each $a_1, a_2 \in \mathcal{A}$ and each nonnegative integer n.

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