NOETHERIANNESS AND LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R, and M an R-module. We show that, whenever $M = \cup_{t>0} (0 :_M \mathfrak{a}^t)$, M is Noetherian if and only if there exists a submodule N of M such that the R-modules $M/\mathfrak{a}N$ and $\cap_{t>0} \mathfrak{a}^t N$ are Noetherian. By using this result, we establish Noetherian properties for local cohomology modules $H^n_{\mathfrak{a}}(M)$ in several cases. For instance, we obtain a new version of Lichtenbaum-Hartshorne Vanishing Theorem.

1. INTRODUCTION

Throughout this paper, R denotes a commutative Noetherian ring with non-zero identity, M is an R-module and $\mathfrak{a}$ is an ideal of R. Also, we use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers.

Local cohomology was first defined and studied by Grothendieck [4]. There are several papers devoted to problems on local cohomology modules (see [6], [7], [10], [11], [13] and [8] for some results in this direction). One of the important problems concerning local cohomology modules is to determine conditions on the R-module M, the ideal $\mathfrak{a}$ and the non-negative integer n which ensure that $H^n_{\mathfrak{a}}(M)$ is finitely generated. It is well known that the local cohomology modules $H^n_{\mathfrak{a}}(M)$ are not generally finitely generated for $n \in \mathbb{N}$. This problem has been investigated by several authors and there has been a great deal of work on it. Most of them involved inductive arguments on n.

Recently, in [9], the first present author with Khosh-Ahang showed that, for a fixed non-negative integer n and a finitely generated R-module M, the following conditions are equivalent (see [9, Theorem 3.1]).

(i) $H^n_{\mathfrak{a}}(M)$ is finitely generated.

(ii) $H^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite and $\mathfrak{a} \subseteq \sqrt{a^n_{\mathfrak{a}}(M)}$, where $a^n_{\mathfrak{a}}(M)$ is the annihilator of $H^n_{\mathfrak{a}}(M)$.

Recall that the local cohomology module $H^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite if $\mathrm{Ext}^i_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$ is finitely generated for all $i \in \mathbb{N}_0$ (cf. [5]).

The purpose of this paper is to develop the above result and to present a method for studying the finiteness properties of local cohomology modules. To do this, since

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