LIFTING DERIVATIONS AND $n-$WEAK AMENABILITY OF THE SECOND DUAL OF A BANACH ALGEBRA

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Abstract. We show that for $n \geq 2$, $n-$weak amenability of the second dual $\mathcal{A}^{**}$ of a Banach algebra $\mathcal{A}$ implies that of $\mathcal{A}$. We also provide a positive answer for the case $n = 1$, which sharpens some older results. Our method of proof also provides a unified approach to give short proofs for some known results in the case where $n = 1$.

The concept of n-weak amenability was initiated and intensively developed by Dales, Ghahramani and Gronbæk [3]. A Banach algebra $\mathcal{A}$ is said to be n-weakly amenable ($n \in \mathbb{N}$) if every (bounded) derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$ (the n^{th} dual of $\mathcal{A}$) is inner. Trivially, 1-weak amenability is nothing else than weak amenability, which was first introduced and intensively studied by Bade, Curtis and Dales [2] for commutative Banach algebras and then by Johnson [9] for a general Banach algebra.

We equip the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$ with its first Arens product and focus on the following question which is of special interest, especially for the case when $n = 1$.

Does $n-$weakly amenability of $\mathcal{A}^{**}$ force $\mathcal{A}$ to be $n-$weakly amenable?

In the present paper first we shall show:

Theorem 1. *The answer to the above question is positive for any $n \geq 2$.*

Then we consider the case $n = 1$, which is a long-standing open problem with a slightly different feature from that of $n \geq 2$. This case has been investigated and partially answered by many authors (see Theorem 6, in which we rearrange some known answers from [5, 6, 7, 8]). As a consequence of our general method of proof (for the case $n = 1$), we present the next positive answer; in which, π denotes the product of $\mathcal{A}$, $\pi^* : \mathcal{A}^* \times \mathcal{A} \to \mathcal{A}^*$ is defined by

$$\langle \pi^*(a^*, a), b \rangle = \langle a^*, \pi(a, b) \rangle, \quad (a^* \in \mathcal{A}^*, a, b \in \mathcal{A}),$$

and $Z_\ell(\pi^*)$ is the left topological centre of π^*, (see the next section).

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